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Research Article

Fractional Ostrowski–Mercer-Type Inequalities for Coordinated s-Convex Functions with Applications to Cubature

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Abstract

A fractional Ostrowsky–Mercer-type approach to function of two variables on the rectangular domain is considered, with special attention paid to the class of functions which are coordinated in the second sense (s)-convex. Fractional integral operators and generalized convexity are used in order to examine estimates for fractional integral deviations based on the mixed partial derivatives of the functions that are dealt with. Given appropriate differentiability and integrability conditions, the proposed framework aims to derived bounds that involve values of the absolute mixed partial derivative at the four corners of the rectangle. The coefficients in these estimates are in terms of Beta and Gamma functions of the fractional kernels. Besides, inequalities of the type extending the scope of the estimates are generalized inequalities involving the absolute mixed partial derivative of (q)-th power. Potential applications of the resulting inequalities to cubature formulas for a double integral involving double, local and composite error estimation are also discussed. The following applications are examples of how fractional integral inequalities and assumptions of coordinated convexity can play a role in an analysis of numerical integration techniques. In the proposed framework fractional calculus, generalized convexity, and numerical analysis are combined together and are the frame of future study of fractional Ostrowski–Mercer-type inequalities and their applications.

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Introduction

The integral inequalities are widely used in mathematical analysis with important applications in approximation theory, numerical integration and the approximation of definite integrals. Of these, the classical Ostrowski inequality, introduced by Ostrowski [1] gives an upper bound for the deviation of a differentiable function from integral mean. Later, Mercer [2] proved a rather important variant of Jensen's inequality, which led to further advances in the study of Mercer-type inequalities. Earlier, these results have been extended to functions of two variables (coordinated convexity) and various inequalities for double integrals have been established [3,4].

The idea of s-convexity has been taken up with much interest by inequality theory [5,6] as a generalisation of classical convexity. There are several variants of Ostrowski-type inequalities under s-convexity assumptions [7]. Moreover, the use of fractional calculus (in the sense of Riemann–Liouville and other fractional integral operators) has led to new methods to generalize the classical integral inequalities [8,9]. The above developments have created useful links between the convexity theory and fractional integral estimates.

The case of two variables has been more convenient for finding fractional Ostrowski-type inequalities [10,11] for functions with this property. The inequalities also give theoretical justifications for the estimation of the approximation errors when it comes to numerical integration and cubature formulas [12]. But the use of a Mercer-type formulation and coordinated s-convexity in conjunction with fractional integral operators is still an interesting field of study.

Research Objective

The objective of this study is to establish fractional Ostrowski–Mercer-type inequalities for coordinated s-convex functions and derive corresponding error estimates with applications to cubature formulas.

Preliminaries and Basic Definitions

In this section, we give the basic definitions, notations and auxiliary results needed to prove fractional Ostrowski–Mercer-type inequalities for coordinatewise convex functions. We first call to mind concepts of convexity and s-convexity, then coordinated convexity and Mercer-type inequalities. Then we present the Riemann–Liouville fractional integral operators and some auxiliary mathematical identities which will be useful for arriving at the main results.

Convex and s-Convex Functions

We begin by recalling the classical definition of convex functions.

Definition 2.1. A function f: I → ℝ, where I is an interval in ℝ, is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

for all x, y ∈ I and λ ∈ [0, 1].

Convexity plays a fundamental role in the theory of mathematical inequalities. Several generalizations of this concept have been introduced to obtain more comprehensive integral inequalities. One such generalization is the notion of s-convexity.

Definition 2.2. A nonnegative function f: I ⊆ [0, ∞) → [0, ∞) is said to be s-convex in the second sense, where s ∈ (0, 1], if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y),$$

for all x, y ∈ I and λ ∈ [0, 1].

It is evident that, when s = 1, the above definition reduces to the classical definition of convexity. Thus, s-convex functions in the second sense constitute a generalized class of convex functions.

The s-convexity condition provides a useful framework for establishing refined bounds involving integral means, fractional integrals, and approximation errors. In particular, it allows the derivation of inequalities under weaker structural assumptions than those imposed by classical convexity.

Coordinated Convex and Coordinated s-Convex Functions

Let $\Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2$ be a rectangular domain, where a < b and c < d. A function defined on Δ is said to be convex on the coordinates if it satisfies the convexity condition separately with respect to each variable.

Definition 2.3. A function f: Δ → ℝ is said to be coordinate-wise convex if, for every fixed y ∈ [c, d], the mapping x ↦ f(x, y) is convex on [a, b], and for every fixed x ∈ [a, b], the mapping y ↦ f(x, y) is convex on [c, d].

Equivalently, for all $x_1, x_2 \in [a, b]$, $y_1, y_2 \in [c, d]$, and λ, μ ∈ [0, 1], the following inequality holds:

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2, \mu y_1 + (1-\mu)y_2) \\ \leq \lambda\mu f(x_1, y_1) + \lambda(1-\mu)f(x_1, y_2) + (1-\lambda)\mu f(x_2, y_1) + (1-\lambda)(1-\mu)f(x_2, y_2). \end{aligned}$$

The concept of coordinated convexity can be extended to s-convex functions as follows.

Definition 2.4. A nonnegative function f: $\Delta \to [0, \infty)$ is said to be coordinated s-convex in the second sense if, for every fixed $y \in [c, d]$, the mapping $x \mapsto f(x, y)$ is s-convex on [a, b], and for every fixed $x \in [a, b]$, the mapping $y \mapsto f(x, y)$ is s-convex on [c, d], where $s \in (0, 1]$.

Consequently, for all $x_1, x_2 \in [a, b]$, $y_1, y_2 \in [c, d]$, and $\lambda, \mu \in [0, 1]$, we have

$$f(\lambda x_1 + (1-\lambda)x_2, \mu y_1 + (1-\mu)y_2) \le \lambda^s \mu^s f(x_1, y_1) + \lambda^s (1-\mu)^s f(x_1, y_2) + (1-\lambda)^s \mu^s f(x_2, y_1) + (1-\lambda)^s (1-\mu)^s f(x_2, y_2).$$

In particular, when s = 1, coordinated s-convexity reduces to coordinated convexity. This property will be used to obtain bounds for the partial derivatives of functions defined on rectangular domains.

Mercer-Type Inequalities

Mercer-type inequalities represent an important extension of classical Jensen-type inequalities and provide useful estimates for convex functions. The following result recalls the weighted form of Mercer's inequality.

Definition 2.5 (Mercer-type inequality). Let f: [a, b] $\to \mathbb{R}$ be a convex function, and let $x_1, x_2, \ldots, x_n \in [a, b]$. Suppose that $\lambda_i \ge 0$, for i = 1, 2, …, n, with

$$\sum_{i=1}^{n} \lambda_i = 1.$$

Then the Mercer-type inequality is given by

$$f\left(a + b - \sum_{i=1}^{n} \lambda_i x_i\right) \le f(a) + f(b) - \sum_{i=1}^{n} \lambda_i f(x_i).$$

The above inequality provides an upper bound involving the endpoint values of a convex function and its weighted interior values. It serves as a useful tool in establishing refinements and generalizations of classical integral inequalities.

For functions of two variables, Mercer-type inequalities can be considered coordinatewise by fixing one variable and applying the corresponding one-dimensional inequality to the other variable. This approach is particularly useful for functions possessing coordinated convexity or coordinated s-convexity properties.

In the present investigation, Mercer-type formulations are considered in conjunction with fractional integral operators to establish new Ostrowski-type estimates on rectangular domains.

Fractional Integral Operators

Fractional calculus provides a natural extension of classical integration and differentiation to non-integer orders. Among the various fractional integral operators, the Riemann–Liouville fractional integrals are widely employed in the study of integral inequalities.

Definition 2.6. Let $f \in L_1([a, b])$ and $\alpha > 0$. The left-sided and right-sided Riemann–Liouville fractional integrals of order α are defined, respectively, by

$$J_{a+}^{\alpha} f(x) = [1/\Gamma(\alpha)] \int_a^x (x-t)^{(\alpha-1)} f(t)\, dt, \quad x > a,$$

$$J_{\beta-}^{\alpha} f(x) = [1/\Gamma(\alpha)] \int_x^b (t-x)^{(\alpha-1)} f(t)\, dt, \quad x < b,$$

where $\Gamma(\alpha)$ denotes the Gamma function, defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{(\alpha-1)} e^{(-t)}\, dt, \quad \alpha > 0.$$

For functions of two variables, the corresponding fractional integral operators can be introduced as follows.

Definition 2.7. Let $f \in L_1(\Delta)$, where $\Delta = [a, b] \times [c, d]$, and let $\alpha, \beta > 0$. The left-sided Riemann–Liouville fractional integral in both variables is defined by

$$J_{a+,c+}^{(\alpha,\beta)} f(x, y) = 1/[\Gamma(\alpha)\Gamma(\beta)] \times \int_a^x \int_c^y (x-u)^{(\alpha-1)} (y-v)^{(\beta-1)} f(u, v)\, dv\, du.$$

Similarly, the right-sided fractional integral is given by

$$J_{\beta-,d-}^{(\alpha,\beta)} f(x, y) = 1/[\Gamma(\alpha)\Gamma(\beta)] \times \int_x^b \int_y^d (u-x)^{(\alpha-1)} (v-y)^{(\beta-1)} f(u, v)\, dv\, du.$$

The mixed left-right fractional integrals are defined by

$$J_{a+,d-}^{(\alpha,\beta)} f(x, y) = \frac{1}{[\Gamma(\alpha)\Gamma(\beta)]} \times \int_a^x \int_y^d (x-u)^{(\alpha-1)} (v-y)^{(\beta-1)} f(u, v)\, dv\, du,$$

and

$$J_{\beta-,c+}^{(\alpha,\beta)} f(x, y) = \frac{1}{[\Gamma(\alpha)\Gamma(\beta)]} \times \int_x^b \int_c^y (u-x)^{(\alpha-1)} (y-v)^{(\beta-1)} f(u, v)\, dv\, du.$$

These operators provide a convenient framework for extending classical double-integral inequalities to fractional settings. In particular, their weighted integral representations are useful in deriving estimates involving partial derivatives and coordinated s-convex functions.

Auxiliary Lemmas and Mathematical Identities

For the development of the main results, we recall some standard identities involving the Beta and Gamma functions.

Lemma 2.1. For $\alpha, \beta > 0$, the Beta function is defined by

$$B(\alpha, \beta) = \int_0^1 t^{(\alpha-1)} (1-t)^{(\beta-1)}\, dt,$$

and satisfies the relation

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Lemma 2.2. For $\alpha > 0$ and $m > -1$, the following

identity holds:

$$\int_0^1 t^{\wedge}(\alpha-1)(1-t)^{\wedge}m\,dt = B(\alpha, m+1) \backslash = \frac{\Gamma(\alpha)\Gamma(m+1)}{\Gamma(\alpha+m+1)}.$$

Lemma 2.3. Let $\alpha > 0$ and $m > -1$. For $h > 0$, we have $\int_0^h t^{\wedge}(\alpha-1)(h-t)^{\wedge}m\,dt = h^{\wedge}(\alpha+m)\,B(\alpha, m+1)$.

Proof. By applying the substitution $t = hu$, we obtain

$$\int_0^h t^{\wedge}(\alpha-1)(h-t)^{\wedge}m\,dt \backslash = h^{\wedge}(\alpha+m)\int_0^1 u^{\wedge}(\alpha-1)(1-u)^{\wedge}m\,du \backslash = h^{\wedge}(\alpha+m)\,B(\alpha, m+1).$$

This completes the proof.

The following elementary identity will also be useful in the derivation of fractional Ostrowski-type estimates:

$$\int_0^1 t^p(1-t)^q dt = B(p+1, q+1), \quad p, q > -1.$$

Moreover, for any integrable function g on [a, b], the change of variables

$$x = a + t(b-a), \quad 0 \le t \le 1,$$

gives

$$\int_a^b g(x)\,dx = (b-a)\int_0^1 g(a+t(b-a))\,dt.$$

Similarly, for an integrable function f on the rectangular domain Δ, we have

$$\int_a^b \int_c^d f(x,y)\,dy\,dx \backslash = (b-a)(d-c)\int_0^1\int_0^1 f(a+t(b-a), c+r(d-c))\,dr\,dt.$$

The preceding definitions and identities provide the necessary mathematical foundation for the subsequent investigation. In particular, the properties of coordinated s-convex functions, together with the fractional integral representations and Beta–Gamma identities, will be employed to establish the principal fractional Ostrowski–Mercer-type inequalities and their corresponding applications to cubature formulas.

Fractional Ostrowski–Mercer-Type Inequalities

The inequalities in this section are the main fractional Ostrowski–Mercer type inequalities for the functions defined on a rectangular domain and satisfying the conditions of the coordinated s-convexity. Using the fractional integral operators introduced in Section 2 we first obtain an auxiliary identity relating to the difference between a suitable weighted combination of the values of the function and the fractional integral means of the function to the mixed partial derivative of the underlying function. This identity is used to get the major inequalities. Then, appropriate bounds are set, assuming that the mixed partial derivative is coordinated s-convex. The estimates obtained are expressed in terms of the parameters of the fractional integral operators, size of the rectangle, and values at the endpoints of the mixed partial derivative. Moreover, several special cases and corollaries are also given to portray the relationship between the results found and the inequalities available in the literature.

Editorial and Mathematical Clarification

The source draft introduces the symbols $E_{(\alpha,\beta)}$, $K_{(\alpha,\beta)}$, ξ, and η abstractly in Lemma 3.1, but does not define them from a specific fractional operator. Thus, Theorems 3.1 and 4.1-4.3 are if-then statements, pending the specification of that identity and kernel. These sections should not be omitted from the manuscript nor should the claims be presented as an Ostrowski–Mercer theorem even if it has been fully described without the missing derivation. Similarly, the discussion of cubature is dependent upon the provision of a concrete rule and an identity of its remainder. The Mercer inequality brought up in Section 2 is background, the draft does not yet extract a separated Mercer transformed fractional remainder. These differences are explicitly expressed because they are not mathematically proved.

Main Fractional Ostrowski–Mercer Identity

In this subsection, we establish the fundamental identity required for deriving the main fractional Ostrowski–Mercer-type inequalities. The identity connects the values of a differentiable function at selected points of a rectangular domain with its corresponding fractional integral expressions.

Let $f: \Delta \to \mathbb{R}$, where $\Delta = [a, b] \times [c, d]$, be a function whose mixed partial derivative exists and satisfies the necessary integrability conditions. Let $\alpha, \beta > 0$ denote the orders of the fractional integral operators.

The principal identity will be formulated in terms of the fractional integral operators defined in Section 2. In particular, the difference between the relevant function values and fractional integral terms will be represented through an appropriate weighted integral involving the mixed partial derivative

$$\frac{\partial^2 f(x,y)}{(\partial x\,\partial y)}$$

This identity provides the analytical foundation for the subsequent estimates.

Lemma 3.1. (Main fractional Ostrowski–Mercer identity)

Let $f: \Delta \to \mathbb{R}$ be sufficiently smooth, with an integrable mixed partial derivative on Δ. Under the

prescribed fractional integral assumptions, the following identity holds:

$$\mathcal{E}_(\alpha,\beta)(f) = \int_0^1 \int_0^1 K _(\alpha,\beta)(t,r)\,[\partial^2 f(\xi(t),\eta(r))/\partial x\partial y]\,dr\,dt,$$

where $\mathcal{E}_(\alpha,\beta)(f)$ denotes the corresponding fractional Ostrowski–Mercer difference, $K_(\alpha,\beta)(t,r)$ is the associated fractional kernel, and ξ(t) and η(r) are suitable affine transformations of the variables.

Proof. The identity is established by applying integration by parts successively with respect to the two variables, followed by the appropriate changes of variables. The boundary terms are combined with the fractional integral expressions to obtain the required representation.

Fractional Ostrowski–Mercer Inequality for Coordinated s-Convex Functions

We now derive the principal inequality by applying the coordinated s-convexity property to the mixed partial derivative appearing in Lemma 3.1.

The following theorem provides the main estimate under the coordinated s-convexity assumption.

Theorem 3.1. Let f: $\Delta \to \mathbb{R}$ be a twice partially differentiable function such that its mixed partial derivative is integrable on Δ. Suppose that

$$|\partial^2 f(x,y)/\partial x\partial y|$$

is coordinate s-convex on Δ, where s ∈ (0, 1]. Then the corresponding fractional Ostrowski–Mercer difference satisfies an upper bound determined by the fractional kernel and the endpoint values of the absolute mixed partial derivative.

Proof. By Lemma 3.1 and the triangle inequality, we obtain

$$|\mathcal{E}_(\alpha,\beta)(f)| \le \int_0^1 \int_0^1 |K _(\alpha,\beta)(t,r)|\,|\partial^2 f(\xi(t),\eta(r))/\partial x\partial y|\,dr\,dt.$$

Since the absolute value of the mixed partial derivative is coordinated s-convex, it can be bounded in terms of its values at the four corners of the rectangular domain. Substituting the corresponding s-convexity estimate into the preceding expression and evaluating the resulting weighted integrals yields the desired upper bound.

Special Cases and Particular Inequalities

The general results obtained in the preceding subsection provide several particular inequalities by assigning suitable values to the fractional parameters and the convexity parameter.

In particular, the following reductions are of interest:

1. Classical convexity case: Setting s = 1 reduces coordinated s-convexity to coordinated convexity.
2. Classical integral case: Taking the fractional orders to their appropriate integer-order limits recovers the corresponding classical integral formulation.
3. Symmetric fractional case: Choosing equal fractional orders, $\alpha = \beta$, yields a symmetric fractional estimate.
4. One-dimensional reduction: By fixing one variable or considering a suitable limiting case, the corresponding one-dimensional Ostrowski-type inequalities can be recovered.

These reductions demonstrate the generality of the established framework and provide connections with several known results in integral inequality theory.

Corollaries for Different Classes of Convex Functions

Several useful corollaries can be obtained from the principal theorem by imposing additional assumptions on the mixed partial derivative.

Corollary 3.1. (Coordinated convex case)

If the absolute value of the mixed partial derivative is coordinated convex, the principal inequality follows by taking s = 1 in Theorem 3.1.

Corollary 3.2. (Bounded mixed partial derivative)

If there exists a constant M > 0 such that

$$|\partial^2 f(x,y)/\partial x\partial y| \le M$$

for all $(x,y) \in \Delta$, then the fractional Ostrowski–Mercer difference admits an upper bound obtained by replacing the mixed partial derivative in the principal estimate with M.

Corollary 3.3. (Constant mixed partial derivative)

If the mixed partial derivative is constant on Δ, the corresponding fractional estimate simplifies to an expression involving only the fractional kernel and the dimensions of the rectangular domain.

The above corollaries illustrate how the general fractional Ostrowski–Mercer inequality can be adapted to different regularity and convexity assumptions. They also provide a basis for obtaining further estimates and applications to numerical integration and cubature formulas.

Mathematical note for manuscript development

In this section the identity and the theorem are given in the form of a publication style framework, but not as complete results. Fractional kernel, the exact definition of $\mathcal{E}_(\alpha,\beta)(f)$, the affine transformations and the resulting constants need to be specified and derived from the exact

fractional operators used in the manuscript. All of these details should be filled in prior to submission, and no numbers or unverifiable identities have been added.

Error Estimates and Generalizations

We compute several error estimates and generalisations of the fractional Ostrowski–Mercer inequalities obtained in Section 3. The main aim is to get explicit upper bounds for the corresponding fractional integral differences, as expressed in terms of the mixed partial derivatives of the underlying functions. The concept of coordinated s-convexity is highlighted in determining these bounds. By using the main fractional Ostrowski–Mercer identity and employing appropriate integral inequalities, estimates relating to the values of the absolute mixed partial derivatives at the four corners of the rectangular domain are derived. In addition, the resulting kernel integrals are conveniently expressed in analytical form by using the Beta and Gamma functions. Finally, generalized inequalities are derived with further integrability and boundedness conditions. The results are used to give a wider background in the study of fractional integral inequalities and applications in numerical approximation.

Upper Bounds for Fractional Integral Differences

We first establish an upper bound for the fractional Ostrowski–Mercer difference by using the integral representation obtained in Lemma 3.1.

Let \\$\alpha = [a,b]\times$[c,d]\, and suppose that \f:\Delta\rightarrow\mathbb R\ satisfies the differentiability and integrability conditions stated in Section 3. Assume that

$$\mathcal E_{\alpha,\beta}(f) = \int_0^1\int_0^1 K_{\alpha,\beta}(t,r)\, f_{xy}(\xi(t),\eta(r))\, dr\, dt,$$

where

$$\xi(t) = a + t(b-a), \quad \eta(r) = c + r(d-c),$$

and $K_{\alpha,\beta}$ denotes the corresponding fractional kernel.

The following theorem provides an upper estimate in terms of the endpoint values of the mixed partial derivative.

Theorem 4.1. Let \f:\Delta\rightarrow\mathbb R\ be twice partially differentiable, with an integrable mixed partial derivative. Suppose that \\lvert f_{xy}\rvert\ is coordinated \s\-convex on \\Delta\, where \s\in(0,1]\. Then:

$$\begin{aligned}|\mathcal E_{\alpha,\beta}(f)| &\le A_{00}|f_{xy}(a,c)| + A_{01}|f_{xy}(a,d)| \\ &\quad + A_{10}|f_{xy}(b,c)| + A_{11}|f_{xy}(b,d)|,\end{aligned}$$

where the coefficients are defined by

$$\begin{aligned}A_{00} &= \int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\,(1-t)^s(1-r)^s\, dr\, dt, \\ A_{01} &= \int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\,(1-t)^s r^s\, dr\, dt, \\ A_{10} &= \int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\, t^s(1-r)^s\, dr\, dt, \\ A_{11} &= \int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\, t^s r^s\, dr\, dt.\end{aligned}$$

Proof. By the identity established in Lemma 3.1 and the triangle inequality, we obtain

$$\begin{aligned}|\mathcal E_{\alpha,\beta}(f)| &\le \int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\, |f_{xy}(\xi(t),\eta(r))|\, dr\, dt.\end{aligned}$$

Since $|f_{xy}|$ is coordinated s-convex, its defining property gives

$$|f_{xy}(\xi(t),\eta(r))| \le t^s r^s |f_{xy}(b,d)| + t^s(1-r)^s |f_{xy}(b,c)| + (1-t)^s r^s |f_{xy}(a,d)| + (1-t)^s(1-r)^s |f_{xy}(a,c)|.$$

Substituting this estimate into the preceding integral inequality and separating the four terms, we obtain

$$\begin{aligned}|\mathcal E_{\alpha,\beta}(f)| &\le |f_{xy}(a,c)| \int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\,(1-t)^s(1-r)^s\, dr\, dt \\ &\quad + |f_{xy}(a,d)| \int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\,(1-t)^s r^s\, dr\, dt \\ &\quad + |f_{xy}(b,c)|\end{aligned}$$

$$\&\#x20;\int_0^1\int_0^1\ \&\#x20;|K_{\α,\β}(t,r)|\ \&\#x20;t^s(1-r)^s\,dr\,dt\\\\\ \&\#x20;\&+|f_{xy}(b,d)|\ \&\#x20;\int_0^1\int_0^1\ \&\#x20;|K_{\α,\β}(t,r)|\ \&\#x20;t^sr^s\,dr\,dt.\ \&\#x20;\end{aligned}$$

Using the definitions of \A_{00},A_{01},A_{10}\, and \A_{11}\, the desired inequality follows. This completes the proof.

Theorem 4.1 establishes a general upper bound for the fractional Ostrowski–Mercer difference. Its main advantage is that the estimate depends only on the values of the absolute mixed partial derivative at the four vertices of the rectangular domain and on the corresponding fractional kernel moments. Consequently, it can be adapted to different fractional integral operators by evaluating the associated coefficients.

Inequalities Involving Beta and Gamma Functions

The coefficients appearing in Theorem 4.1 can often be evaluated explicitly by employing the classical Beta and Gamma functions. These special functions provide an effective means of simplifying the fractional kernel integrals and obtaining computable constants in the resulting inequalities.

Recall that, for \p,q>0\, the Beta function is defined by

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,dt,$$

and satisfies

B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.

We first present an auxiliary result that will be used in evaluating the coefficients of the principal inequality.

Lemma 4.1. Let \\(p,q>0\\) and \\(s\in(0,1]\\). Then

$$\int_0^1 t^{p-1}(1-t)^{q+s-1}\,dt = B(p,q+s), \quad \text{and} \quad \int_0^1 t^{p+s-1}(1-t)^{q-1}\,dt = B(p+s,q).$$

$$\text{Consequently, } B(p,q+s)=\frac{\Gamma(p)\Gamma(q+s)}{\Gamma(p+q+s)} \text{ and } B(p+s,q)=\frac{\Gamma(p+s)\Gamma(q)}{\Gamma(p+q+s)}.$$

Proof. The results follow directly from the definition of the Beta function. Applying the standard relationship between the Beta and Gamma functions yields the required representations.

This completes the proof.

Theorem 4.2. Under the assumptions of Theorem 4.1, suppose that the absolute value of the fractional kernel has the separable representation

$$|K_{\α,\β}(t,r)|=C_{\α,\β}\,t^{p-1}(1-t)^{q-1}\,r^{u-1}(1-r)^{v-1},$$

where $C_{\α,\β}>0$ *and* $p,q,u,v>0$. *Then*

$$\begin{aligned}|\mathcal{E}_{\α,\β}(f)| &\le C_{\α,\β}\big[B(p+s,q)B(u+s,v)|f_{xy}(b,d)| \\ &\quad + B(p+s,q)B(u,v+s)|f_{xy}(b,c)| \\ &\quad + B(p,q+s)B(u+s,v)|f_{xy}(a,d)| \\ &\quad + B(p,q+s)B(u,v+s)|f_{xy}(a,c)|\big].\end{aligned}$$

Proof. By Theorem 4.1, the required estimate is obtained by evaluating the four coefficients separately.

For the first coefficient, we have

$$A_{00}=C_{\α,\β}\int_0^1 t^{p-1}(1-t)^{q+s-1}\,dt\times\int_0^1 r^{u-1}(1-r)^{v+s-1}\,dr = C_{\α,\β}\,B(p,q+s)B(u,v+s).$$

Similarly,

$$A_{01}=C_{\α,\β}\,B(p,q+s)B(u+s,v),$$

$$A_{10}=C_{\α,\β}\,B(p+s,q)B(u,v+s),$$

and

$$A_{11}=C_{\α,\β}\,B(p+s,q)B(u+s,v).$$

Substituting these expressions into Theorem 4.1 gives the desired result. This completes the proof.

Theorem 4.2 demonstrates how the fractional Ostrowski–Mercer inequality can be expressed in

terms of Beta functions and, equivalently, Gamma functions. Such representations are particularly useful when the fractional kernel consists of products of power functions. They also facilitate the comparison of different fractional integral operators and the investigation of limiting cases.

Generalized Fractional Ostrowski–Mercer Inequalities

In this subsection, we extend the preceding results by incorporating additional assumptions on the mixed partial derivative. In particular, we establish a generalized estimate involving the \L^q\-type behavior of the derivative.

Theorem 4.3. Let \\(f:\Delta\rightarrow\mathbb R\\) satisfy the assumptions of Lemma 3.1. Suppose that

$$|f_{xy}|^q \text{ is coordinated } s\text{-convex on } \Delta, \text{ where } q \geq 1 \text{ and } s\in(0,1]. \text{ Then}$$

$$\begin{aligned}
&|\mathcal{E}_{\alpha,\beta}(f)| \\
&\leq \left(\int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\, dr\, dt\right)^{1-\frac{1}{q}} \\
&\times \left[\int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\, |f_{xy}(\xi(t),\eta(r))|^q\, dr\, dt\right]^{\frac{1}{q}}.
\end{aligned}$$

Moreover, if

$$\begin{aligned}
B_{00} &= \int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\, (1-t)^s(1-r)^s\, dr\, dt, \\
B_{01} &= \int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\, (1-t)^s r^s\, dr\, dt, \\
B_{10} &= \int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\, t^s(1-r)^s\, dr\, dt, \\
B_{11} &= \int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\, t^s r^s\, dr\, dt,
\end{aligned}$$

then

$$\begin{aligned}
&|\mathcal{E}_{\alpha,\beta}(f)| \\
&\leq \left(\int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\, dr\, dt\right)^{1-\frac{1}{q}} \\
&\times \Big[B_{00}|f_{xy}(a,c)|^q + B_{01}|f_{xy}(a,d)|^q \\
&\quad + B_{10}|f_{xy}(b,c)|^q + B_{11}|f_{xy}(b,d)|^q\Big]^{\frac{1}{q}}.
\end{aligned}$$

Proof. Applying Hölder's inequality in its weighted integral form, we obtain

$$\begin{aligned}
|\mathcal{E}_{\alpha,\beta}(f)| &\leq \left(\int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\, dr\, dt\right)^{1-\frac{1}{q}} \\
&\times \left[\int_0^1\int_0^1 |K_{\alpha,\beta}(t,r)|\, |f_{xy}(\xi(t),\eta(r))|^q\, dr\, dt\right]^{\frac{1}{q}}.
\end{aligned}$$

Since $|f_{xy}|^q$ is coordinated s-convex, it follows that

$$\begin{aligned}
|f_{xy}(\xi(t),\eta(r))|^q &\leq (1-t)^s(1-r)^s|f_{xy}(a,c)|^q \\
&+ (1-t)^s r^s|f_{xy}(a,d)|^q \\
&+ t^s(1-r)^s|f_{xy}(b,c)|^q \\
&+ t^s r^s|f_{xy}(b,d)|^q.
\end{aligned}$$

Substituting this inequality into the preceding estimate and evaluating the resulting weighted integrals in terms of \B_{00},B_{01},B_{10}\, and \B_{11}\, we obtain the required result. This completes the proof.

Theorem 4.3 provides a broader estimate than Theorem 4.1 by allowing the use of higher powers of the absolute mixed partial derivative. In particular, it is useful when the \q\-th power of the derivative possesses a suitable coordinated convexity property.

Corollary 4.1. (Coordinated convexity case). If \s=1\, then the generalized inequality in Theorem 4.3 reduces to the corresponding fractional Ostrowski–Mercer inequality for functions whose \q\-th power of the absolute mixed partial derivative is coordinated convex.

Corollary 4.2. (Bounded mixed partial derivative). Suppose that there exists a constant \\(M>0\\) such that

$$|f_{xy}(x,y)| \leq M, \quad (x,y)\in\Delta. \text{ Then}$$

$$|\mathcal{E}_{\alpha,\beta}(f)| \leq M \int_0^1 \int_0^1 |K_{\alpha,\beta}(t,r)|\, dr\, dt.$$

Proof. The result follows immediately by substituting the assumed bound into the integral representation in Lemma 3.1 and applying the triangle inequality.

Corollary 4.3. (Equal fractional orders). If the fractional orders are equal, that is, $\alpha=\beta=\gamma>0$, then Theorems 4.1–4.3 yield the corresponding symmetric-order fractional Ostrowski–Mercer estimates, provided that the associated kernel satisfies the required integrability conditions.

The preceding results establish a collection of generalized fractional Ostrowski–Mercer inequalities under different assumptions on the mixed partial derivative. The estimates involving Beta and Gamma functions provide explicit representations whenever the fractional kernel admits a suitable power-function decomposition. These findings extend the applicability of the principal inequality and provide a mathematical basis for deriving error estimates in numerical integration and cubature formulas.

Applications to Numerical Integration and Cubature Formulas

The inequalities in the previous sections can be used to make estimates on the error involved in numerical approximations to double integrals. More specifically, the condition of the absolute mixed partial derivative to be s-convex enables the error to be bounded using values of the derivatives at the vertices of a rectangular region. This section outlines some of the ways these numbers can be used in cubature analysis. By first requiring the exact cubature rule and functional of the remainder to be matched to the explicit fractional identity in Lemma 3.1, the remainder is then sought in a known form. The exact cubature rule and the functional of the remainder, however, are matched with the explicit fractional identity given in Lemma 3.1, after which the remainder is sought in a known form.

Construction of Cubature Formulas

Let $\Delta = [a,b] \times [c,d]$ be a rectangular region and let f be an integrable function on Δ. A cubature formula approximates the double integral of f over Δ by a finite combination of function values at selected points. For a partition $a = x_0 < x_1 < \cdots < x_m = b$ and $c = y_0 < y_1 < \cdots < y_n = d$, define the subrectangles $\Delta_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$. A composite cubature approximation can be written in the general form

$$Q_{m,n}(f) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} Q_{ij}(f),$$

where $Q_{ij}(f)$ is a local approximation to the integral of f over Δ_{ij}. Depending on the selected nodes and weights, Q_{ij} may represent a midpoint, corner-based, trapezoidal, or another rectangular cubature rule. The associated local error is

$$R_{ij}(f) = \iint_{\Delta_{ij}} f(x,y)\, dy\, dx - Q_{ij}(f),$$

and the total error satisfies $R_{m,n}(f) = \sum_{ij} R_{ij}(f)$. To apply the fractional Ostrowski–Mercer identity, the local rule must be chosen so that its remainder can be represented by the fractional difference functional and kernel introduced in Section 3. The nodes, weights, and normalization constants therefore cannot be specified independently of that identity.

Error Bounds for Cubature Formulas

Suppose that, on each subrectangle Δ_{ij}, the local remainder admits an identity of the form

$$R_{ij}(f) = \iint_{[0,1]^2} K_{ij}(t,r)\, f_{xy}(\xi_i(t), \eta_j(r))\, dr\, dt,$$

where $\xi_i(t) = x_i + t(x_{i+1} - x_i), \eta_j(r) = y_j + r(y_{j+1} - y_j)$, and K_{ij} is the kernel determined by the selected fractional cubature rule. If $|f_{xy}|$ is coordinated s-convex on Δ_{ij}, the endpoint estimates from Theorem 4.1 yield

$$\begin{aligned}|R_{ij}(f)| \leq\ & A_{00}^{(ij)} |f_{xy}(x_i, y_j)| + A_{01}^{(ij)} |f_{xy}(x_i, y_{j+1})| \\ & + A_{10}^{(ij)} |f_{xy}(x_{i+1}, y_j)| \\ & + A_{11}^{(ij)} |f_{xy}(x_{i+1}, y_{j+1})|,\end{aligned}$$

where the coefficients $A_{kl}^{(ij)}$ are the corresponding weighted moments of $|K_{ij}|$. By the triangle inequality, the composite error is bounded by

$$|R_{m,n}(f)| \leq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |R_{ij}(f)|.$$

Consequently, a computable global bound follows once the local kernels and their moments have been evaluated. If the kernel has a separable power form, the moments may be expressed through Beta functions as in Theorem 4.2. This gives explicit constants in terms of the fractional parameters and the side lengths of the subrectangles. The formula remains conditional on the exact kernel and remainder identity being specified and verified.

Applications to Double Integrals

Consider the double integral $I(f) = \iint_{\Delta} f(x,y)\, dy\, dx$. A change of variables x = a + t(b − a) and y = c + r(d − c) maps Δ onto the unit square and gives

$$I(f) = (b-a)(d-c) \iint_{[0,1]^2} f(a + t(b-a), c + r(d-c))\, dr\, dt.$$

This representation helps to develop product-type quadrature and cubature rules for rectangles. If the mixed derivative exists and is coordinate-wise s-

convex then the estimates from Sections 3 and 4 can be used to ensure that a rule with a remainder having the above fractional-kernel representation controls the error. The estimate obtained is sensitive to the four corner values of $|f_{xy}|$ and the moments of the kernels over the whole unit square. Discrete bounds can be helpful for the evaluation of numerical integration methods for smooth functions, as long as the assumptions of the relevant theorem are met.

Special Cases and Numerical Illustration

Several limiting or special cases clarify the scope of the estimates. First, setting s = 1 reduces coordinated s-convexity to coordinated convexity, and the endpoint estimates take the corresponding convex form. Second, if $|f_{xy}|$ is bounded on Δ by a constant M, then any valid kernel representation immediately gives $|R(f)| \leq M \iint _[0,1]^2 |K(t,r)| \, dr$ dt. Third, when the kernel is separable, K (t, r) = $K_1(t)K_2(r)$, its absolute moment factors into the product of one-dimensional moments, which may simplify evaluation through Beta and Gamma functions.

A numerical illustration should be included after the fractional remainder functional has been explicitly defined. A suitable test function may be selected, its exact double integral calculated analytically, and the proposed cubature approximation evaluated for stated nodes and weights. The absolute error should then be compared with the theoretical upper bound obtained from the verified kernel moments. No numerical values are reported here because the current formulation does not yet specify the complete kernel, the exact functional $E\alpha,\beta(f)$, or a fully defined cubature rule. Supplying numerical results before these elements are fixed would risk presenting unsupported constants or an invalid comparison.

Comparison and Discussion

Comparison with Existing Ostrowski-Type Inequalities

Ostrowski-type inequalities have a range of applications in mathematical analysis, such as approximation theory, numerical integration, and error estimation in integration. The classical Ostrowski's inequality (3.11) [13] is a bound for the difference between a differentiable function and its integral mean based on the knowledge of its derivative. Another related but different development was proposed by Mercer (2003) [14] who introduced a variation of Jensen inequality with endpoint values and weighted interior values. The present investigation is motivated by these results, which give two foundations of relevance for the present study, namely: estimates of integral deviations based on derivatives and Mercer inequalities for convex functions.

The present manuscript is an attempt to bring together these thoughts in the context of a fractional Ostrowski-Mercer type framework of functions of two variables defined on a rectangular domain. The proposed method is different from the classical one-dimensional Ostrowski inequality as it is expressed in terms of the mixed partial derivative f_xy and fractional integral operators. The desired estimate is thus multidimensional and is related to the particular fractional remainder identity used in the manuscript. Clear relations with the classical Ostrowski and Mercer inequalities need to be established, which involves deriving the identity and identifying its kernel and boundary terms.

An additional difference lies with the convexity assumption. Hudzik and Maligranda (1994) talked about properties of s-convex functions and considered the first and second meanings of this generalized convexity [15]. In this manuscript, the absolute value of the mixed partial derivative is used for coordinated s-convexity which is defined as follows.In the present manuscript the absolute value of the mixed partial derivative is used for the coordinated s-convexity as follows. If this is so, then the absolute value of f_xy at an interior point of the rectangle is bounded by the absolute values of f_xy at the four corners of the rectangle. This results in an endpoint based estimate, if the fractional identity and its kernel are carefully determined.

One-dimensional settings of fractional Ostrowski inequalities have been also studied. For instance, Alomari et al. (2012) took mappings whose derivatives were in the s-convex sense using fractional integral operators [16]. The present study's work is very similar to theirs, considering the integration of fractions, and generalized convexity. The present manuscript takes into account a rectangular domain and the coordinated behaviour of a mixed partial derivative, which is different to what is done in the other manuscript. This comparison serves to point out the extension that was intended, but is not proof of the newness or validity of the proposed inequality.

Sarikaya et al. (2014) introduced Ostrowski-type

inequalities for coordinate-wise convex functions [17] for functions of two variables. They give a relevant basis for the estimates on rectangular domains and for the application of coordinatewise convexity in double-integral inequalities. In this manuscript, we intend to develop this line further by adding fractional integral operators, a Mercer-type formulation and coordinated s-convexity. Specifically, if s = 1, then coordinated s-convexity becomes coordinated convexity. To determine whether the resulting inequality is a particular result of Sarikaya et al. one must compare the full identities, assumptions, and constants.

In the paper Budak and co-authors [18] studied Ostrowski-type inequalities for double integrals using fractional integral operators. This is relevant work since it illustrates the application of fractional techniques to two-dimensional integral estimates. The present manuscript, on the other hand, also treats fractional operators on a rectangle, but plans to define the difference in a Mercer-type fashion and to use the four corners values of |f_xy| to bound the difference. The proposed endpoint coefficients are given in terms of weighted moments of the fractional kernel and for appropriate separable kernels can be written in the form of Beta functions. They characterize the structure of the manuscript; their validity is dependent on the explicit definition and derivation of the kernel.

Budak et al. (2021) introduced generalized Ostrowski-, Simpson-, and trapezoidal-type inequalities for coordinated convex functions based on generalized fractional integrals [19]. Their results provide examples of how classical inequalities can emerge in the context of a generalized fractional framework with the right choice of parameters. For this reason, the special cases in the present manuscript should be demonstrated directly in the expression; the one of "coordinated convexity" at s = 1; and the other of the "classical integral formulations" in the limiting cases of the fractional parameters. Conclusions should not be made without reviewing the kernel, normalization factors and boundary terms.

Dragomir et al. [20] also constructed an Ostrowski-type inequality for double integrals using $L^{p\text{-norms}}$ and applied this with respect to numerical integration. The norm-based approach employs complete information about the derivative in question while the endpoint-based estimate suggested here is designed to make use of the values of the derivative at the four corners of a rectangle. These are all ways to provide information and may be applicable in various data circumstances. A comparison of their sharpness or computational performance would have to compare both bounds with the same functions and domains.

Reduction to Classical Inequalities

One of the main strengths of a generalized inequality is that the answer to the question: What can be recovered from the inequality under specific values of the parameters? can be answered. In this context, taking s = 1 makes the second sense of coordinated s-convexity coincide with coordinated convexity. This provides a direct link to the coordinated-convex framework analysed by Sarikaya et al. (2014). A simplified special case should be derived by making the substitution s = 1 in the proven estimate, and simplifying the coefficients [17]. The fractional order α and β also need to be taken into account. In order to recover a classical integral expression, the fractional operators must converge to the desired integer-order or ordinary-integral expression. This reduction should be checked by checking the definitions in the manuscript including limiting behaviour of the kernel, normalization constants and boundary terms. The significance of considering special cases in fractional inequality frameworks is illustrated in earlier works, such as those of Alomari et al. (2012) and Budak et al. (2021) [16, 19]. The Mercer-type component calls for an additional explanation. The classical Ostrowski inequality measures the difference between a function and the integral mean of the function, while Mercer's inequality is a type of that inequality with endpoint and weighted interior values. The manuscript must then indicate precisely what the Mercer-type expression goes into the fractional remainder functional. The derivation needs to be very rigorous to avoid being confused with existing fractional Ostrowski inequalities and to support the contribution that is claimed as a result of combining these ideas.

Advantages and Potential Applications

The proposed framework is intended for achieving its combination of fractional integral operators and coordinated s-convexity of functions defined on rectangular domains. The estimate can be written in terms of the values of the absolute mixed partial derivative if the latter is s-convex and the fractional remainder identity holds true in the rectangle

where the four corners of the rectangle are the points. In cases where derivative information is known at the edge of a computational domain, such an endpoint-based bound might be appropriate. Coefficients expressed as Beta and Gamma functions are also discussed in the manuscript. This representation is possible if the absolute fractional kernel is a valid representation with a separable form that is power function. The following general conditional expression, described in Theorem 4.2, and the following estimate involving the q-th power of the absolute mixed partial derivative, described in Theorem 4.3, give expressions of this type. Although these results may give analytical forms of the constants, the applicability of these depends upon checking that the actual kernel does satisfy the above mentioned assumptions. One possible application is to attack the error analysis of numerical integration, specifically cubature formulas for double integrals. An endpoint estimate can be used on each subrectangle if a local cubature remainder can be shown to be represented by the fractional kernel in the main identity. The resulting local bounds can then be added together to give an estimate of the error for a composite rule. This is in line with the applications of Ostrowski-type inequalities presented by Dragomir et al. [20] and fractional estimates for double integrals by Budak et al. [18]. This application is not yet considered established, and some steps must be taken before it can be considered such. The kernel and the functional difference expression must be first explicitly defined, as well as the affine transformations. The identity in Lemma 3.1 should then be determined from the given fractional operators, with a careful attention to all the boundary terms and constants. These coefficients should be computed with the same kernel as used in Theorems 4.1 and 4.2. Lastly, a particular cubature rule and its remainder terms should be introduced and checked. It would be helpful to illustrate the numerical error that actually occurs by using test functions and to compare the numerical error with the theory bounds.

The literature cited in this manuscript indicates that Ostrowski type inequalities have been extended in various convexity conditions, fractional integral operators and multidimensional settings. The basic results are due to Ostrowski (1938) and Mercer (2003) [13-15] describe the mathematical properties of the s-convexity. Alomari et al. (2012), Sarikaya et al. (2014), Budak et al. (2019), Budak et al. (2021), and Dragomir et al. (2003), provide further studies on fractional integral inequality, coordinated convexity, double integral and numerical approximation [16-19]. In this context, the main objective of the present study is to develop fractional Ostrowsky–Mercer type inequalities for coordinated s-convex functions and investigate their application in the estimation of cubature errors. The endpoint estimates and the two forms of the Beta-function representations and generalization of the q-type are proposed extensions of the current methodology. The key contribution of the manuscript, however, is only substantiated once the fractional difference and kernel are explicitly defined and the corresponding constants and limiting cases are verified, and once the main identity is rigorously proved. A full application (cubature treatment) and numerical validation would further make the results understandable and useful.

Conclusion

We have studied a fractional Ostrowski-Mercer type framework for functions of two variables on a rectangular domain, focusing on the definition of coordinated (s)-convexity and its possible numerical integration applications. The study combines some fundamental concepts in inequality theory such as the classical Ostrowski inequality, Mercer inequalities, fractional integral operators and generalized convexity. This combination gives a framework for study of integral estimates in non-classical convex frameworks.

One key part of the approach suggested is the use of the mixed partial derivative of the function being considered. The goal is to try to get estimates taking into account the coordinated (s)-convexity of the absolute value of the function on the rectangle, in terms of the values of the mixed partial derivative at the four vertices of the rectangle. If derivative information is available at the boundary of the region, such endpoint-based estimates could be used as a convenient way to assess integral deviations. Moreover, use of Beta and Gamma functions offers a systematic method of expressing the coefficients associated with proper fractional kernels.

The study also shows the possible connection between fractional Ostrowski–Mercer-type inequality and the estimation of the error in cubature formulas. The framework proposed here,

in particular, can be used to obtain local error estimates on rectangular subdomains and then global error bounds for composite numerical integration schemes. The estimates obtained might prove helpful in the study of approximation schemes for double integrals if the appropriate fractional remainder identities and cubature rules can be carefully derived.

The present study also suggests some directions for future research. Explicit derivation and verification of the fractional Ostrowsky–Mercer identity, as well as the determination of sharp or improved constants, and the study of special cases that recover classical integral inequalities. Other tasks could include the construction of explicit cubature formulas, numerical experimentation with the formulas and the comparison of the theoretical error bounds with the computed approximation errors for some test functions.

Finally, it seems that the use of fractional integral operators and coordinated (s)-convexity is a promising line for further investigation of Ostrowski-type inequalities for functions of two variables. This direction of study can help develop the theory of integral inequalities and its applications in numerical analysis, by establishing a connection between fractional integral deviations and mixed partial derivatives. Analytical verification and numerical validation will be needed to ensure the accuracy of the proposed results and their practical implications.

Declarations

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